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SEISMIC RESPONSE OF HORIZONTAL SOIL LAYERS

By I. M. Idriss,[1] A. M. ASCE, and H. Bolton Seed,[2] M. ASCE

INTRODUCTION

In many cases the ground motions developed near the surface of a soil deposit during an earthquake may be attributed primarily to the upward propagation of shear waves from an underlying rock formation. If the ground surface, the rock surface, or the boundaries between different soil layers are inclined, analyses of the response of the soil deposit can be made only by techniques such as the finite-element method. If the ground surface, the rock surface, and the boundaries between soil layers are essentially horizontal, however, the lateral extent of the deposit has no influence on the response, and the deposit may be considered as a series of semi-infinite layers. In such cases the ground motions induced by a seismic excitation at the base are only the result of shear deformations in the soil, and the deposit may be considered as a one-dimensional shear beam. Methods of analyzing the response of such deposits form the subject of this paper.

The equation of motion for the response at any depth of a semi-infinite soil deposit can be written readily. However, closed-form solutions for these equations can only be derived for a few idealized conditions involving linear elastic materials whose properties vary with depth in a manner which can be represented by a relatively simple mathematical expression. In general, soils do not behave as linear elastic materials and their properties in any soil deposit are likely to vary in an irregular fashion. In such cases it is necessary to resort to numerical techniques to evaluate the response of the deposit to a given base excitation.

Herein closed-form solutions are derived for evaluating the response of soil layers with linearly elastic properties varying in a prescribed manner. A

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lumped-mass solution is presented for the evaluation of the response of soil deposits with linearly elastic but nonuniform material properties. The accuracy and stability of this elastic lumped-mass solution are studied, and a criterion for determining the accuracy of the solution is proposed. A previously developed lumped-mass solution for evaluating the response of a soil deposit with irregularly varying bilinear stress-strain characteristics is then outlined. Finally, a simplified method of analyzing conditions of this type by treating the soils as equivalent linear elastic materials is presented. Typical examples of the results obtained using the various analytical procedures are presented for illustrative purposes.

LINEAR ELASTIC ANALYSES

Closed-Form Solutions

Earthquake Response.—The equation of motion for the vibration of a semi-infinite layer subject at its base (see Fig. 1) to a horizontal seismic motion, u_g, is

$$\rho(y)\frac{\partial^2 u}{\partial t^2} + c(y)\frac{\partial u}{\partial t} - \frac{\partial}{\partial y}\left[G(y)\frac{\partial u}{\partial y}\right] = -\rho(y)\frac{d^2 u_g}{dt^2} \quad \ldots\ldots\ (1)$$

in which $\rho(y)$ = mass density at a depth y; $c(y)$ = viscous damping coefficient at a depth y; $G(y)$ = shear modulus at a depth y; and $u(y,t)$ = relative displacement at a depth y at time t. Considering the layer to be composed of soils that are linearly elastic with uniform density and viscous damping characteristics, and letting the shear modulus variation with depth be prescribed by

$$G = Ky^p \quad \ldots\ldots\ (2)$$

in which K and p are constants, Eq. 1 becomes

$$\rho\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} - \frac{\partial}{\partial y}\left[K y^p \frac{\partial u}{\partial y}\right] = -\rho\ddot{u}_g \quad \ldots\ldots\ (3)$$

Eq. 3 is a second-order hyperbolic partial differential equation. When p = 0 and $\ddot{u}_g$ is a known function of time or is equal to zero, Eq. 3 reduces to a linear hyperbolic partial differential equation whose solution is readily available in standard mathematical books. Eq. 3, with $p = 0$ and $\ddot{u}_g$ = either zero or a known function of time, has been utilized by Kanai (14,15),[3] Mattheisen, et al. (18), Zeevaert (25), Herrera and Rosenblueth (6), Kobayashi and Kagami (16), and others in their studies of the seismic response of soil layers.

The solution of Eq. 3 when $p \neq 0$ may be obtained by the method of separation of variables. Letting

$$u(y,t) = \sum_{n=1}^{\infty} Y_n(y)\, X_n(t) \quad \ldots\ldots\ (4)$$

gives $$Y_n(y) = (\beta_n/2)^b\,\Gamma(1-b)\,(y/H)^{b/\theta} J_{-b}\left[\beta_n (y/H)^{1/\theta}\right] \quad \ldots\ldots\ (5)$$

$$\ddot{X}_n + 2\lambda_n\omega_n\dot{X}_n + \omega_n^2 X_n = -R_n\ddot{u}_g \quad \ldots\ldots\ (6)$$

[3] Numerals in parentheses refer to corresponding items in the Appendix I.—References.

in which J_{-b} = Bessel function of the first kind of order $-b$; β_n = roots of $J_{-b}(\beta_n) = 0, n = 1, 2, \ldots$; $\omega_n = \beta_n\sqrt{K/\rho}/\theta H^{1/\theta}$, H is the total thickness of the layer; $\lambda_n = c/2\rho\omega_n$ is the damping ratio; $R_n = 1/\left[(\beta_n/2)^{1+b}\,\Gamma(1-b)\, J_{1-b}(\beta_n)\right]$; Γ = gamma function; and b and θ are constants related to p by

$$p\theta - \theta + 2b = 0$$
$$p\theta - 2\theta + 2 = 0 \quad \ldots\ldots\ (7)$$

These equations have been derived in more detail elsewhere (11).

It should be noted that the above equations are restricted to $p \leq 1/2$. When $p > 1/2$, a solution in terms of Bessel functions cannot be obtained.

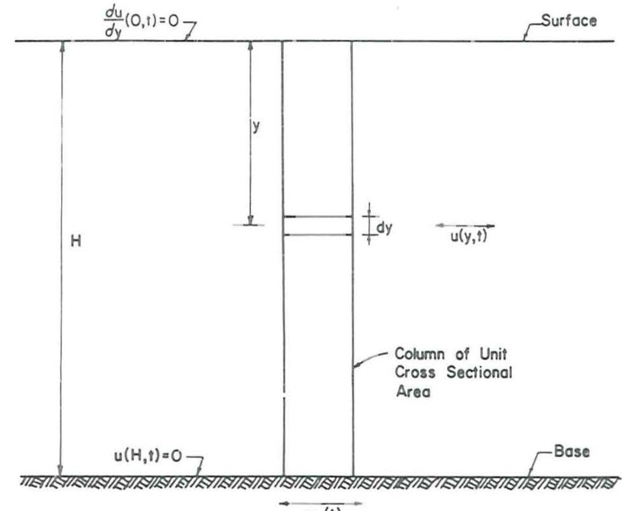

FIG. 1.—CROSS-SECTION AND BOUNDARY CONDITIONS OF A SEMI-INFINITE SOIL LAYER SUBJECTED TO A HORIZONTAL SEISMIC MOTION AT ITS BASE

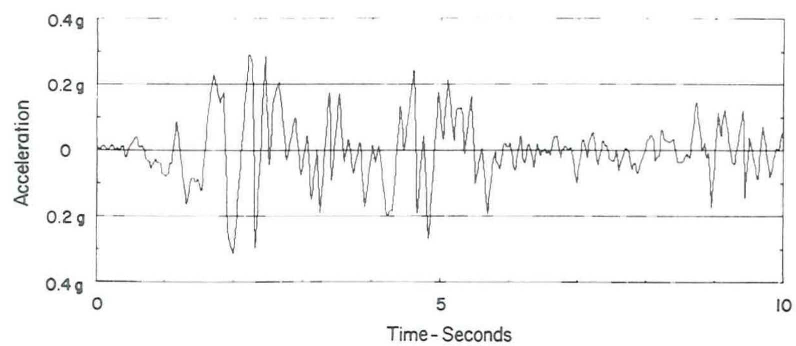

FIG. 2.—ACCELERATION RECORD USED IN ANALYSES (N-S COMPONENT, 1940 EL CENTRO EARTHQUAKE)

Eq. 5 defines the mode shape of the system, $Y_n(y)$, during the n^{th} mode of vibration whose circular frequency is ω_n. The values of $X_n(t)$ may be computed from a solution of Eq. 6 using iterative procedures, such as the one proposed by Newmark (19), or direct numerical procedures such as the step-by-step (1,24), or a Runge-Kutta (7) process. Once the values of $Y_n(y)$ and $X_n(t)$ are determined, the relative displacement at any depth y is given by Eq. 4. The relative velocity, relative acceleration, and strain at any depth y at any instant of time can be obtained by an appropriate differentiation of Eq. 4.

A soil layer composed mainly of cohesive soils might be considered to have a uniform modulus; but for a layer composed mainly of cohesionless soils the modulus will vary with depth. Experimental results (4,5) have shown that the modulus of a cohesionless soil varies with the confining pressure to powers of about 1/3 or 1/2. For illustration purposes, this modulus is considered herein to be proportional to the cube root of depth. A similar relationship was used by Rashid in analyzing the dynamic response of earth dams composed of cohesionless soils (22).

Soil Layers with Uniform Shear Modulus.—For this case $p = 0$, $G = K$, and from Eq. 7, $b = 1/2$ and $\theta = 1$. The equation of motion, Eq. 3, then reduces to

$$\rho \frac{\partial u^2}{\partial t^2} + c \frac{\partial u}{\partial t} - G \frac{\partial^2 u}{\partial y^2} = - \rho \ddot{u}_g$$

which is a standard linear hyperbolic differential equation with constant coefficient whose solution is readily available. However, the solution may also be obtained by substituting $p = 0$, $b = 1/2$, and $\theta = 1$ in Eqs. 5 and 6. Thus

$$Y_n(y) = \cos \frac{(2n - 1)}{2} \frac{y}{H} \quad \ldots\ldots\ldots\ldots \quad (8a)$$

$$\ddot{X}_n + 2\lambda_n \omega_n \dot{X}_n + \omega_n^2 X_n = (-1)^n \frac{4}{(2n - 1)\pi} \ddot{u}_g \quad \ldots\ldots\ldots\ldots \quad (8b)$$

$$\omega_n = \frac{(2n - 1)\pi}{2H} \sqrt{G/\rho} \quad \ldots\ldots\ldots\ldots \quad (8c)$$

Soil Layers with Shear Modulus Proportional to the Cube Root of Depth.—For this case $p = 1/3$, $G = Ky^{1/3}$, $b = 0.4$, and $\theta = 1.2$. Hence

$$Y_n(y) = \Gamma(0.6) \left(\frac{\beta_n}{2}\right)^{0.4} \left(\frac{y}{H}\right)^{1/3} J_{-0.4}\left[\beta_n \left(\frac{y}{H}\right)^{5/6}\right] \quad \ldots\ldots\ldots\ldots \quad (9a)$$

$$\ddot{X}_n + 2\lambda_n \omega_n \dot{X}_n + \omega_n^2 X_n = - \frac{1}{\Gamma(0.6)(\beta_n/2)^{1.4} J_{0.6}(\beta_n)} \ddot{u}_g \quad \ldots\ldots \quad (9b)$$

$$\omega_n = \frac{\beta_n \sqrt{K/\rho}}{1.2\, H^{5/6}} \quad \ldots\ldots\ldots\ldots \quad (9c)$$

in which β_n are the roots of $J_{-0.4}(\beta_n) = 0$, and $\beta_1 = 1.7510$, $\beta_2 = 4.8785$, $\beta_3 = 8.0166$, $\beta_4 = 11.1570$ . . . etc.

FORTRAN IV listing of a computer program to evaluate the seismic response of a semi-infinite layer for the two special cases outlined above has been presented elsewhere (11).

These closed-form solutions may be used to evaluate the response of soil layers (Fig. 3) having the following uniform properties: total thickness, H = 50 ft; unit weight, γ = 120 pcf; elastic modulus, $E = 7 \times 10^5$ psf; Poisson's ratio, $\mu = 0.45$; shear modulus, $G = 2.41 \times 10^5$ psf; and damping ratio, λ =

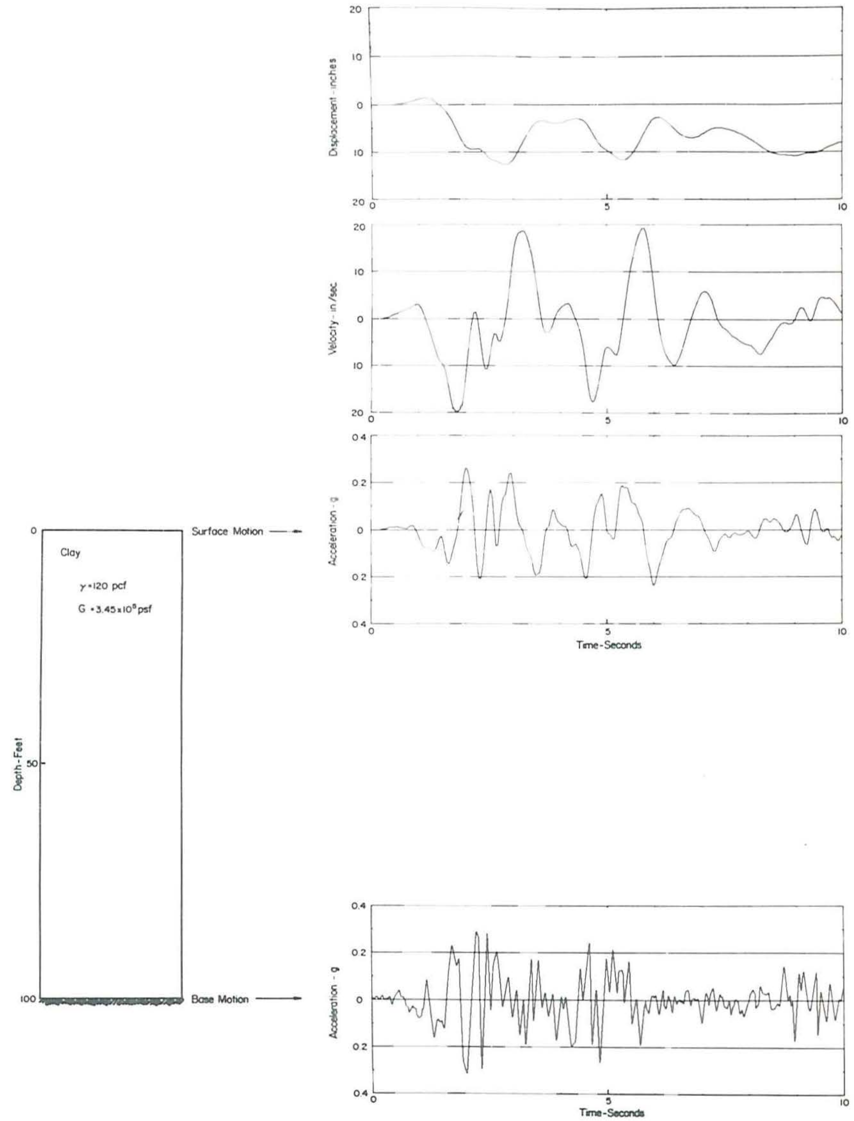

FIG. 3.—SURFACE RESPONSE OF LAYER WITH UNIFORM PROPERTIES

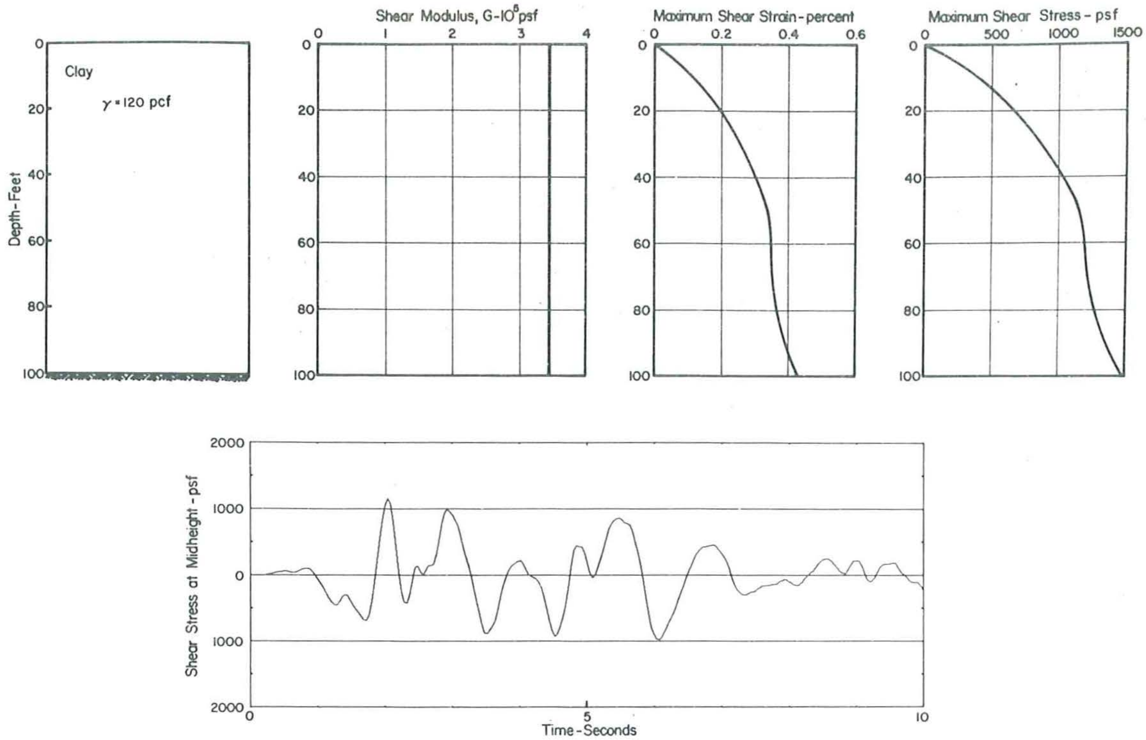

FIG. 4.—STRESSES AND STRAINS DEVELOPED WITHIN LAYER WITH UNIFORM PROPERTIES

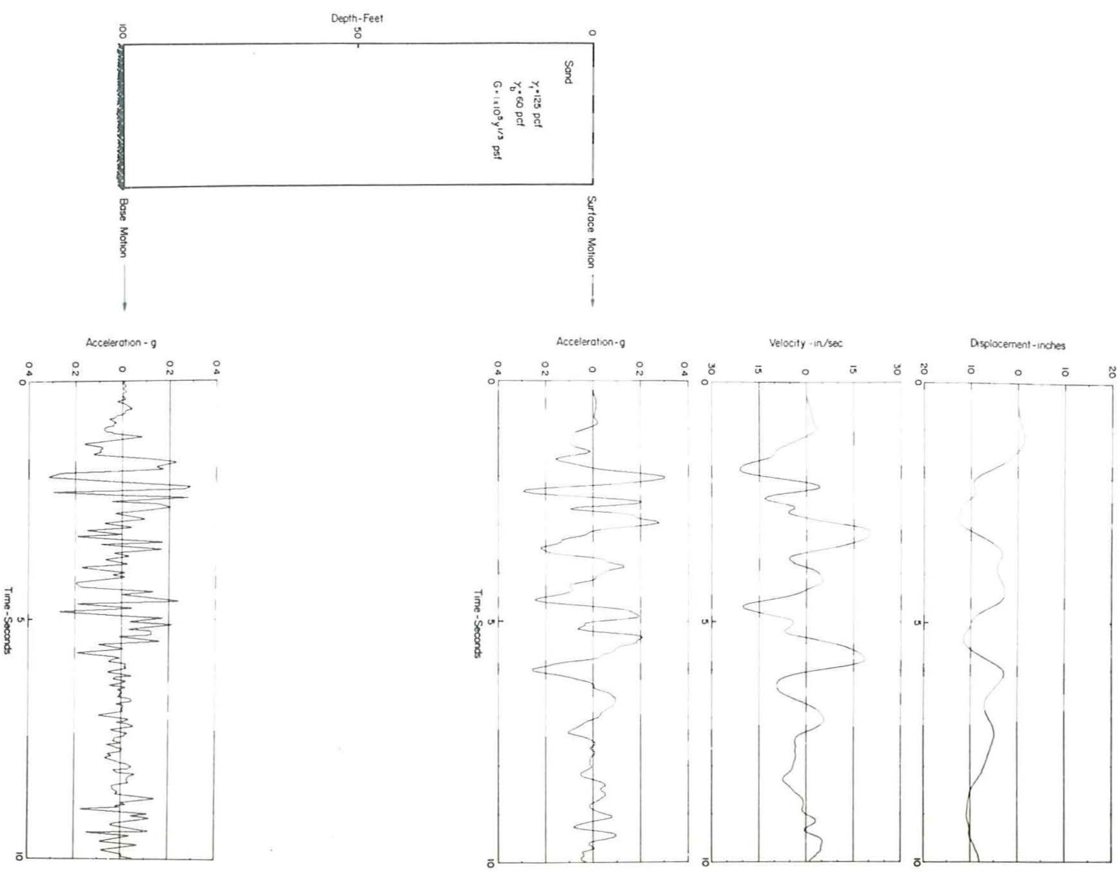

FIG. 5.—SURFACE RESPONSE OF LAYER WITH MODULUS PROPORTIONAL TO CUBE ROOT OF DEPTH

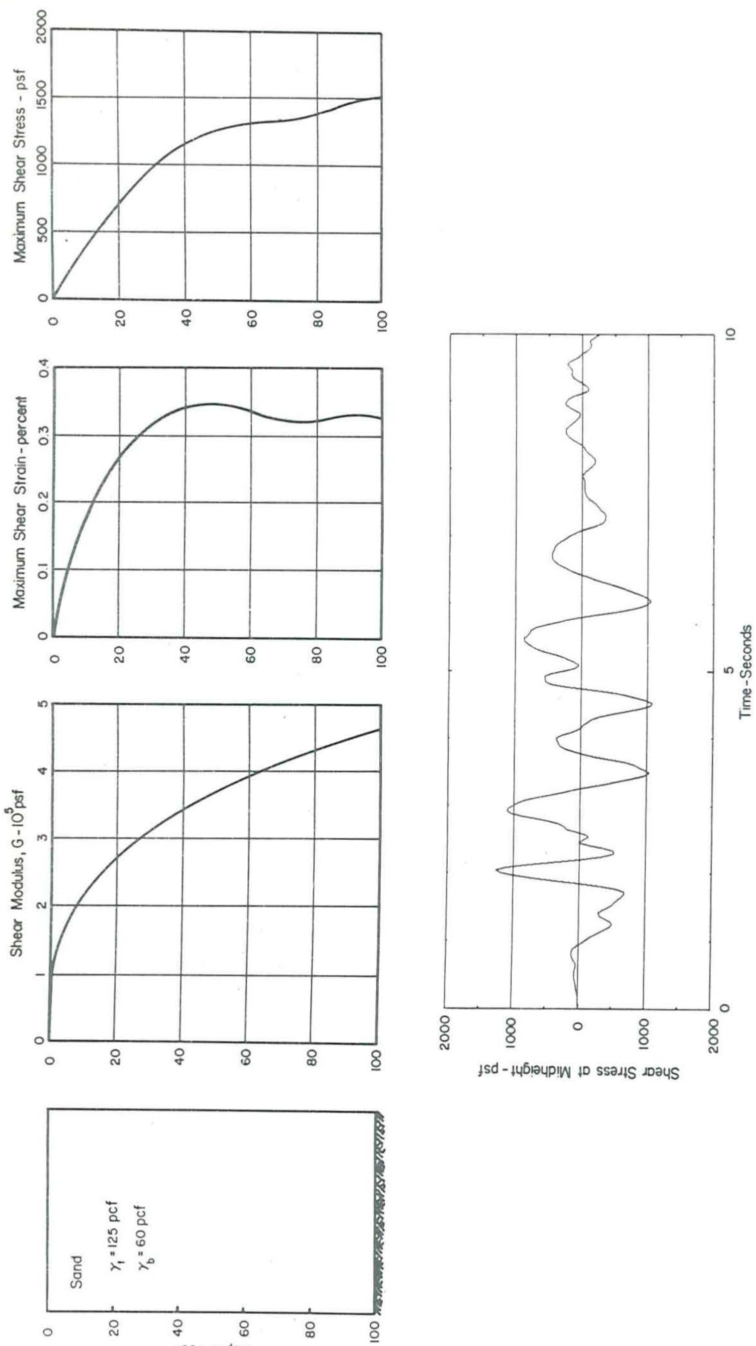

FIG. 6.—STRESSES AND STRAINS DEVELOPED WITHIN LAYER WITH MODULUS PROPORTIONAL TO CUBE ROOT OF DEPTH

0.2 for all modes. The response of this layer to the input base motion shown in Fig. 2 is presented in Figs. 3 and 4. The time histories of surface acceleration, velocity, and displacement are presented in Fig. 3. The maximum values of strain and stress developed throughout the depth of the layer and the time history of shear stress at a depth of 50 ft are shown in Fig. 4.

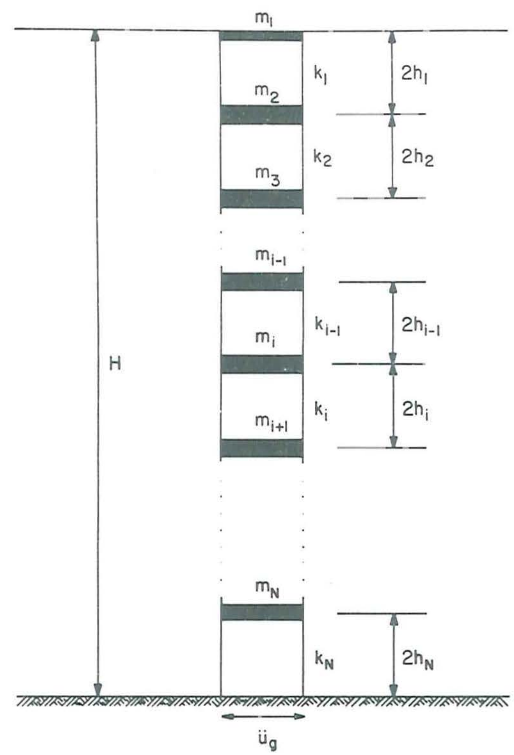

FIG. 7.—LUMPED-MASS IDEALIZATION OF A SEMI-INFINITE LAYER (LINEAR ELASTIC SOLUTION)

The response of a soil layer with modulus proportional to the cube root of depth, to the same input base motion is shown in Figs. 5 and 6. The properties of this layer are: total thickness, H = 100 ft; total unit weight, γ_t = 125 pcf; buoyant unit weight, γ_b = 60 pcf; elastic modulus, $E = 2.5 \times 10^5\ y^{1/3}$ psf; Poisson's ratio, $\mu = 0.25$; shear modulus, $G = 1 \times 10^5\ y^{1/3}$ psf; and damping ratio, λ = 0.2 for all modes. The time histories of surface acceleration, velocity, and displacement determined for the layer are shown in Fig. 5. The maximum values of strain and stress developed throughout the depth of this layer together with the time history of shear stress at a depth of 50 ft are presented in Fig. 6.

Similar solutions can readily be developed for other layers with material properties varying with depth in a regular manner.

Lumped-Mass Solution

Earthquake Response.—To analyze the response of a soil deposit having irregularly varying, but linearly elastic, soil properties, it is necessary to use a lumped-mass type of analysis. The deposit, which may consist of several sublayers of varying properties, is idealized (Fig. 7) by a series of discrete (lumped) masses interconnected by springs that resist lateral deformations. These springs represent the stiffness properties of the material between any two discrete masses. Damping is assumed linearly viscous.

When the deposit is subjected to a horizontal seismic motion through its base, the equation of motion of the system may be represented in matrix form as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{R(t)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

in which $[M]$, $[C]$, and $[K]$ are the mass, viscous damping, and stiffness matrices, respectively, $\{R(t)\}$ is the earthquake load vector, and $\{u\}$ is the relative displacement vector (dots represent differentiation with respect to time). These matrices and vectors are of order N, where N is the number of lumped masses used in idealizing the layer.

Details of the formulation of these matrices and vectors and the method used to solve Eq. 10 are given elsewhere (11). In general, however, once the geometry and material properties of the deposit and the earthquake motion at the base are known, the evaluation of the seismic response of the deposit involves the following steps:

1. The deposit is idealized by subdividing it into N levels. The mass and stiffness matrices of the resulting system are computed from the known geometry and material properties of the layer.

2. The mass and stiffness matrices are used to set up the characteristic value problem of the system, viz.

$$[K]\{\phi^n\} = \omega_n^2 [M]\{\phi^n\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

in which ϕ_i^n is the mode shape at level i during the n^{th} mode of vibration whose circular frequency is ω_n. The solution of Eq. 11 gives the mode shapes and frequencies of the system.

3. The normal equations are solved for the response of each mode at each instant of time and the modal responses are superposed to give the time history of displacement at each level i

$$u_i(t) = \sum_{n-1}^{N} \phi_i^n X_n(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots (12)$$

in which $X_n(t)$ is the normal coordinate for the n^{th} mode. Velocities, accelerations, and strains are obtained by appropriate differentiation of Eq. 12.

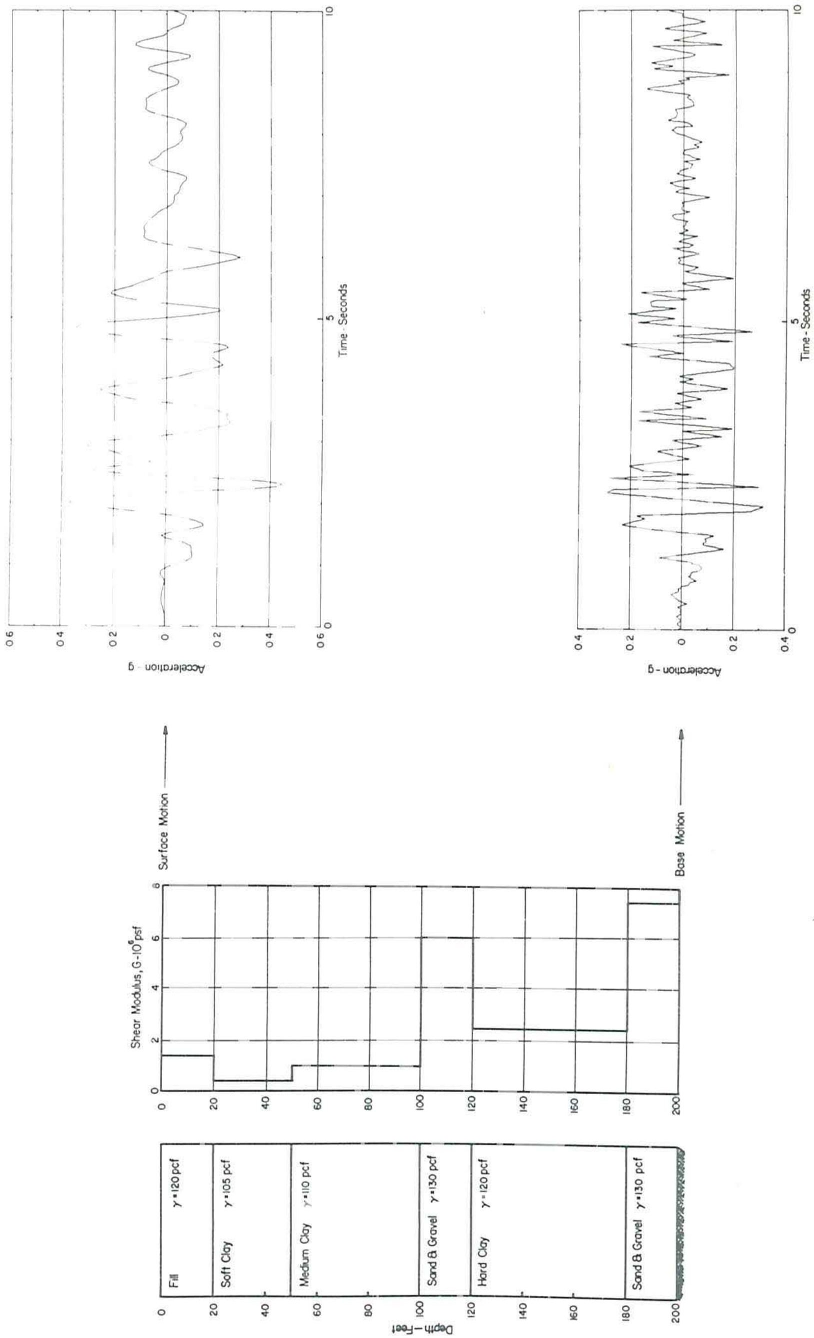

FIG. 8.—SURFACE RESPONSE OF DEPOSIT WITH NONUNIFORM LINEAR ELASTIC PROPERTIES

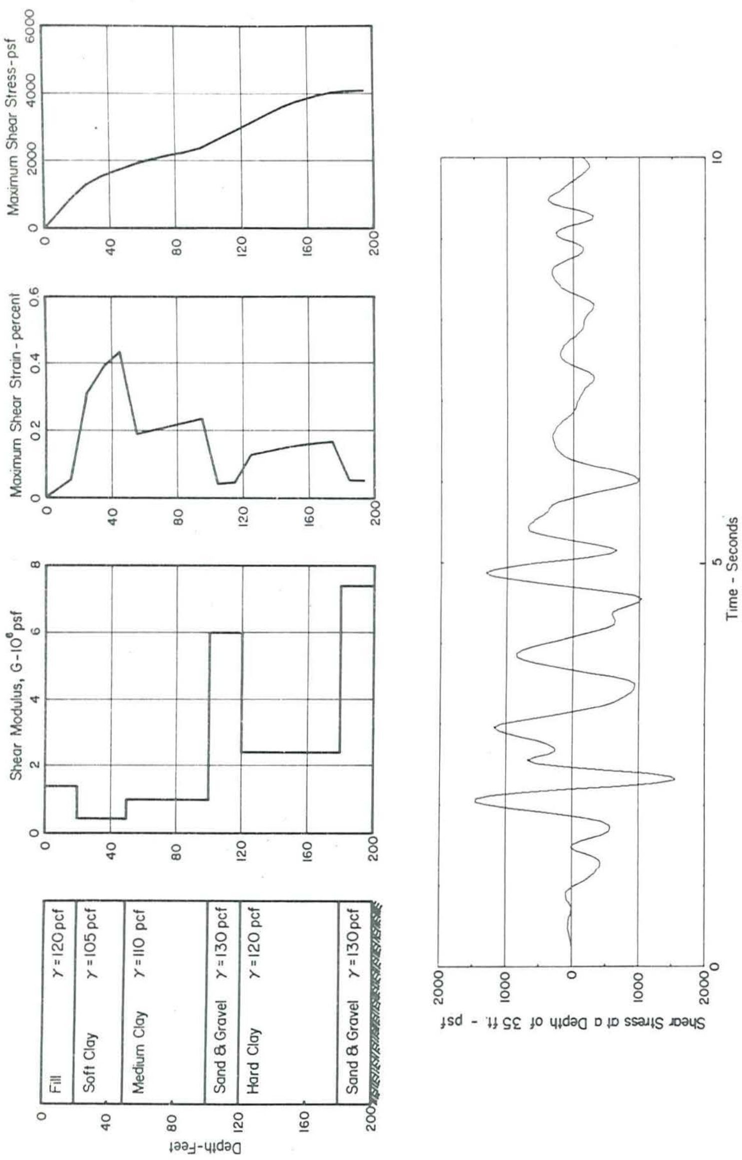

FIG. 9.—STRESSES AND STRAINS DEVELOPED WITHIN DEPOSIT WITH NONUNIFORM LINEAR ELASTIC PROPERTIES

FORTRAN IV listing of a computer program to evaluate the seismic response of a soil deposit by the lumped-mass solution has been presented elsewhere (11).

The lumped-mass analysis may be used to evaluate the response of deposits composed of a number of different layers (Fig. 8). This deposit has a total depth of 200 ft and its mass was lumped at 20 levels for analysis purposes. A damping ratio of 0.2 was used for all modes, and the acceleration record shown in Fig. 2 was used as input base motion. The resulting time history of accelerations at the ground surface is shown in Fig. 8. The maximum values of shear strain and shear stress developed throughout the depth of the deposit and the time history of shear stress at a depth of 35 ft are presented in Fig. 9.

Accuracy and Stability of the Lumped-Mass Solution.—The lumped-mass solution is essentially a finite difference method for the solution of the hyperbolic partial differential equation of motion (Eq. 3). Associated with this method are essential questions of accuracy and stability. The accuracy and stability of the lumped-mass solution has been evaluated (11) for layers composed of uniform material properties and for layers with modulus proportional to the cube root of depth. The accuracy and stability of the lumped-mass representation for these layers were ascertained by performing analyses using varying values of N and comparing the results to those of the closed-form solutions. These studies indicated that the accuracy depends on the number of divisions, N, used, and on the fundamental period, T_1, of the layer as depicted in Fig. 10 where ERS is the percentage error in the lumped-mass representation. The stability of the solution depends on the time interval, Δt, used in integrating the normal equations and the value of the lowest period, T_{NN} (i.e., period of the highest mode of vibration), included in the analysis. Based on these results, proposed criteria for the accuracy and stability of the lumped-mass representation are outlined below.

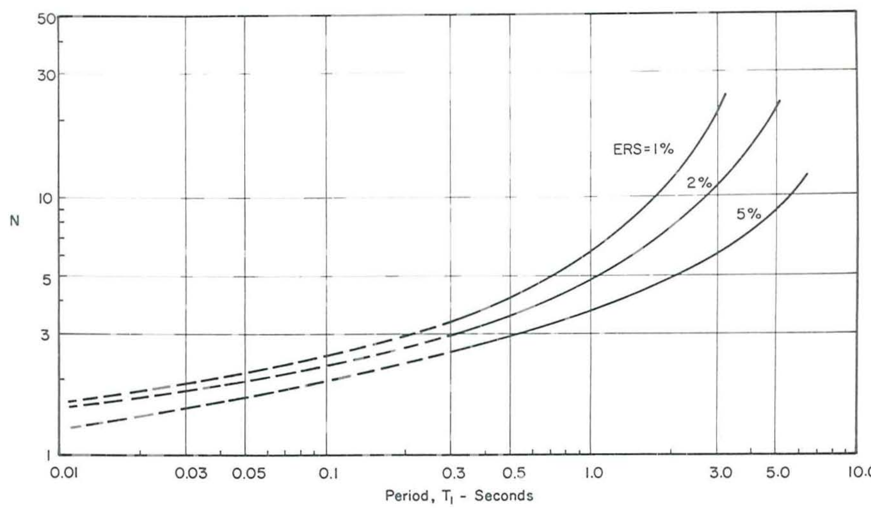

FIG. 10.—N VERSUS T_1 FOR EQUAL VALUES OF ERS

TABLE 1.—CRITERION FOR THE ACCURACY OF THE LUMPED-MASS SOLUTION ILLUSTRATIVE EXAMPLE FOR SOIL DEPOSIT 296 FEET THICK: DETERMINATION OF VALUES OF N_i

Segment thickness, H_i, in feet	Modulus, G_i, in thousands of pounds per square foot	Unit weight, γ_i, in pounds per cubic foot	Period, $(T_1)_i$, in seconds	N_i from Fig. 10	
				ERS < 1%	ERS < 5%
10	280	105	0.137	3	2
10	600	105	0.093	3	2
20	200	120	0.345	4	3
40	143	102	0.754	5	4
20	250	102	0.285	4	3
15	800	126	0.133	3	2
135	357	110	1.67	11	5
45	571	125	0.469	4	3
				ΣN_i = 37	24

TABLE 2.—CRITERION FOR THE ACCURACY OF LUMPED-MASS SOLUTION ILLUSTRATIVE EXAMPLE FOR SOIL DEPOSIT 296 FT THICK: RESULTS OF ANALYSES

ΣN_i From Table 1		N Used in analysis	T_{1N},[a] in seconds	ERS, as a percentage
ERS < 1%	ERS < 5%			
37	24	8	3.60	19
		14	3.53	6
		24	3.51	1.5
		37	3.51	0.3
		46	3.51	—

[a] T_{1N} is the fundamental period of the layer determined from a solution of the characteristic value problem of the lumped-mass system (Eq. 11).

Criterion for the Accuracy of the Lumped-Mass Solution.—The number of divisions, N, to be used in analyzing the response of a soil deposit with any distribution of material properties, may be chosen with the aid of Fig. 10 as follows:

1. The deposit is divided into several segments, each having uniform material properties, and the period $(T_1)_i$ of each segment is computed using Eq. 8c

$$(T_1)_i = \frac{4\,H_i}{\sqrt{G_i\,g/\gamma_i}} \qquad (13)$$

in which H_i is the thickness of the i^{th} segment whose shear modulus is G_i and whose unit weight is γ_i, and g is the acceleration of gravity.

2. Each segment is then divided into N_i levels. The number N_i is obtained from Fig. 10 by entering the figure with the computed value of $(T_1)_i$. The entire deposit is divided into at least N levels when $N = \Sigma N_i$.

This procedure and the adequacy of the proposed criterion were ascertained (11) by analyzing several deposits having nonuniform material properties. The determination of the values of N_i and the results of the analyses for one of these deposits are presented in Tables 1 and 2. The values of N_i for each segment of the deposit were chosen from Fig. 10 for values of ERS < 1% and ERS < 5% and are listed in Table 1. The response of the deposit was then evaluated using values of N less than, equal to, and greater than the values of ΣN_i listed in Table 1, in order to check the adequacy of the criterion. The results of these analyses (Table 2) indicate the adequacy of the proposed criterion.

Criterion for the Stability of the Lumped-Mass Solution.—Analyses of a number of deposits have indicated that the lumped-mass representation remains stable if $T_{NN} \geq 2\Delta t$ when a step-by-step (1,24) analysis procedure is used, and if $T_{NN} \geq 5\Delta t$ when Newmark's (19) iterative procedure is used for the integration of the normal equations.

BILINEAR ANALYSIS

Under conditions of strong ground shaking the stress-strain relationships of most soils have the nonlinear form illustrated in Fig. 11(*a*). In such cases, the stress-strain relationship can be taken into account conveniently in a response analysis by idealizing the curved form for any soil layer by the equivalent bilinear system shown in Fig. 11(*b*).

A lumped-mass solution to evaluate the seismic response of a clay layer with varying bilinear stress-strain characteristics has previously been presented by Parmelee et al. (20). The layer was represented by the model shown in Fig. 12 with each of the linkages, connecting adjacent pairs of masses, consisting of a Kelvin model attached in series to a dashpot. The spring of the Kelvin model was considered to have idealized force-displacement characteristics corresponding to the soil stress-strain characteristics shown in Fig. 11(*b*). The dashpot of the Kelvin model represented viscous damping in the soil. The other dashpot, which is attached in series to the Kelvin model, is considered to represent creep characteristics of the soil.

Test results on silty clay soil samples, reported by Parmelee et al. (20) indicated that the creep dashpot coefficients were large numbers, on the order of 10^5 psf-sec to 10^8 psf-sec, and that these coefficients increased with the strength of the soil. A large creep coefficient results in essentially no displacement across the creep dashpot, i.e., the dashpot is "locked," and the results are unaffected by creep. Consequently, the creep dashpot has been considered locked and no creep effects are included in the analyses.

The change in shear stress, $\Delta \sigma_{xyi}$, developed within segment i of the layer (between masses m_i and m_{i+1}) is given by

$$\Delta \sigma_{xyi} = G_i \, \Delta \epsilon_{xyi} + \beta_i \, \Delta \dot{\epsilon}_{xyi} \quad \text{. . . . . (14)}$$

in which G_i is the shear modulus, β_i is the viscous damping coefficient, and $\Delta \epsilon_{xyi}$ is the change in shear strain (dots represent differentiation with respect to time). The value of G_i to be used in Eq. 14 is equal to either G_{1i} or G_{2i} and is controlled by the yield shear strain ϵ_{yi} for that segment. The moduli G_{1i}

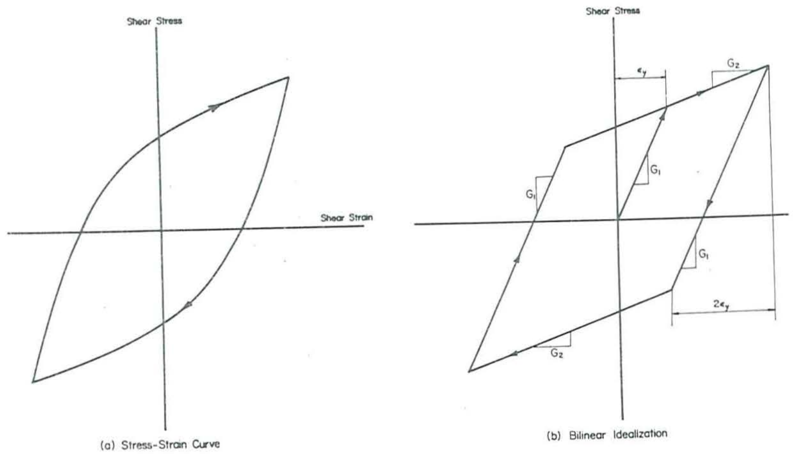

FIG. 11.—STRESS-STRAIN CHARACTERISTICS OF SOIL

and G_{2i} and the yield shear strain, ϵ_{yi}, are determined from appropriate dynamic tests [e.g., Parmelee et al. (20) and Thiers and Seed (23)]. The viscous damping coefficient can also be evaluated from appropriate dynamic tests (20). The variations of all these parameters throughout the depth of the layer are ascertained by performing tests on samples obtained from various depths within the layers.

The derivation of the equations of motion for the lumped-mass representation shown in Fig. 12 has been covered in detail elsewhere (20,21,11). In general, however, the solution proceeds as follows:

1. The geometry and material properties of the layer and the earthquake motion to be applied at the base are ascertained.
2. The layer is divided into N levels and the mass of each level is lumped at the top of the level. (The choice of the number N may be made using the criterion outlined earlier.)

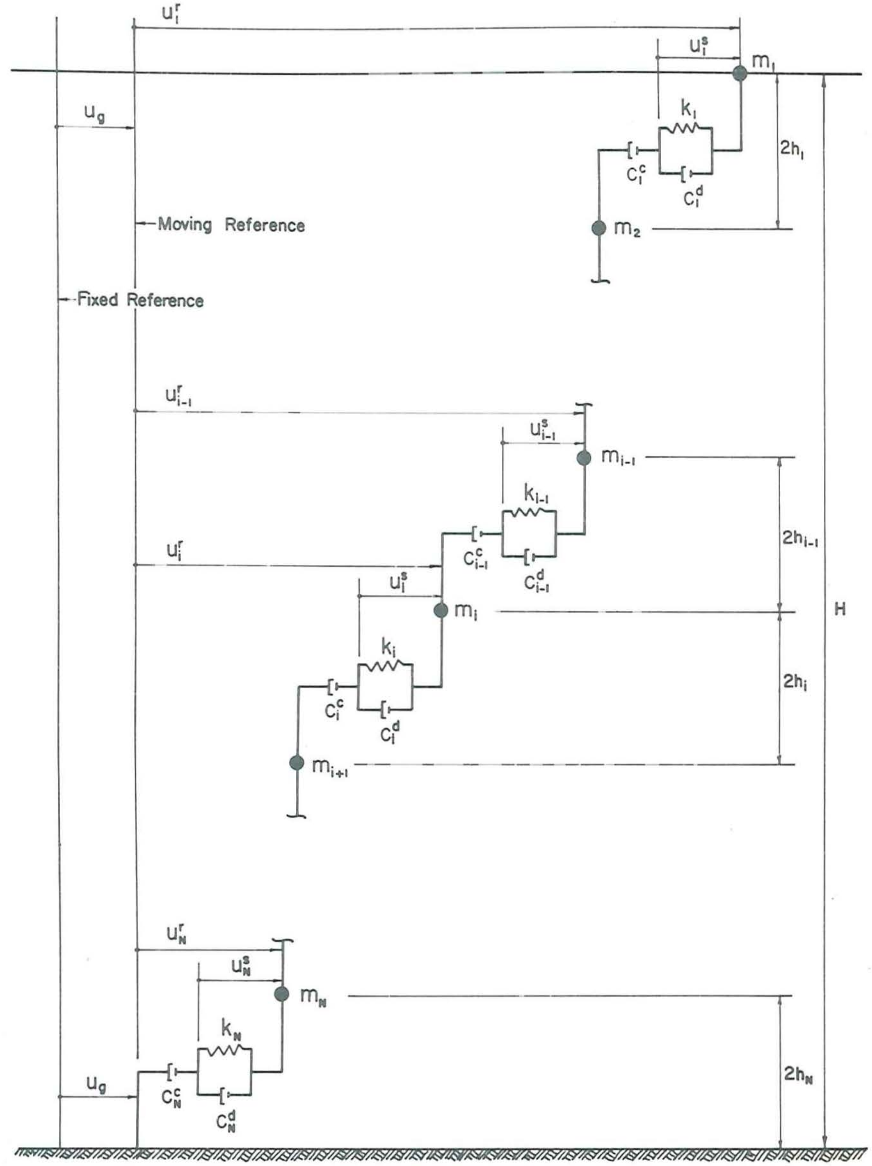

FIG. 12.—LUMPED-MASS IDEALIZATION OF A SEMI-INFINITE LAYER (BILINEAR SOLUTION)

3. The values of k_i and c_i^d (spring stiffness and viscous damping coefficients, respectively, of the Kelvin model connecting masses m_i and m_{i+1}) are determined from the known values of G_i and β_i. This is done by equating the change in force in the Kelvin model to that in the corresponding segment of the layer. The change in force, ΔF_i, in the layer is equal to the change in shear stress (Eq. 14) times the area, which is unity in this case. The change in force in the Kelvin model is given by

$$\Delta F_i = k_i \, \Delta u_i^s + c_i^d \, \Delta \dot{u}_i^s \quad \ldots\ldots\ldots\ldots \quad (15)$$

in which Δu_i^s and $\Delta \dot{u}_i^s$ are the changes in displacement and velocity, respectively, across the Kelvin model. [Note that the displacement, u_i^s, across the Kelvin model is equal to $(u_i^r - u_{i+1}^r)$, in which u_i^r is the relative displacement of mass m_i, when the creep effects are not included.] The values of k_{1i} and k_{2i} are evaluated from Eq. 15 by substituting G_{1i} and G_{2i}, respectively, into Eq. 14. The use of either k_{1i} or k_{2i} in the equations of motion is controlled by the yield displacement, u_{yi}, which is evaluated from the yield shear strain, ϵ_{yi}, by

$$u_{yi} = 2 h_i \, \epsilon_{yi} \quad \ldots\ldots\ldots\ldots \quad (16)$$

4. The equations of motion are set up using the values of m_i, c_i^d, and k_i and are solved using a step-by-step (24) procedure. Initially, the spring stiffness coefficient $k_i = k_{1i}$ is used for each segment of the layer. The behavior, within each segment, continues to be controlled by k_{1i} until a limiting displacement $u_i^s = u_{yi}$ is reached, at which stage further displacements are evaluated using k_{2i}. When the force, F_i, is reversed, behavior is again controlled by k_{1i} until a displacement change equal to $2u_{yi}$ has occurred. Subsequent displacements are determined using k_{2i} until reloading occurs. On reloading, the coefficient k_{1i} is again operative until a change of displacement of $2u_{yi}$ has taken place and the coefficient k_{2i} again controls the behavior. Similar behavior is considered to develop during subsequent cycles.

5. The response values (including accelerations, velocities, displacements, strains and stresses) of each segment of the layer, throughout the duration of the input base motion, are evaluated from the solution of the equations of motion. Appropriate plotting routines are used to plot the time history of any specified response value of any segment of the layer.

Parmelee et al. (20) presented a computer program, written in FORTRAN II, to perform the evaluation of the seismic response of a layer with hysteretic bilinear stress-strain characteristics. For the purpose of this study, the program was modified and rewritten in FORTRAN IV, details of which are given elsewhere (11).

The use of the bilinear lumped-mass solution may be illustrated by obtaining the response of a soil layer having the following properties: total thickness of layer, H = 30 ft; unit weight, γ = 120 pcf; initial (or primary) shear modulus, G_1 = 100,000 psf; secondary shear modulus, G_2 = 40,000 psf; yield shear strain, ϵ_y = 0.1% ; and viscous damping coefficient, β = 20 psf-sec.

For computation purposes the layer was divided into 15 segments. The choice of this number was made in accordance with the criteria proposed earlier in this report. The fundamental period of the layer is equal to 0.733 sec if $G = G_1$ and equal to 1.16 sec when $G = G_2$. The shortest period, T_{NN},

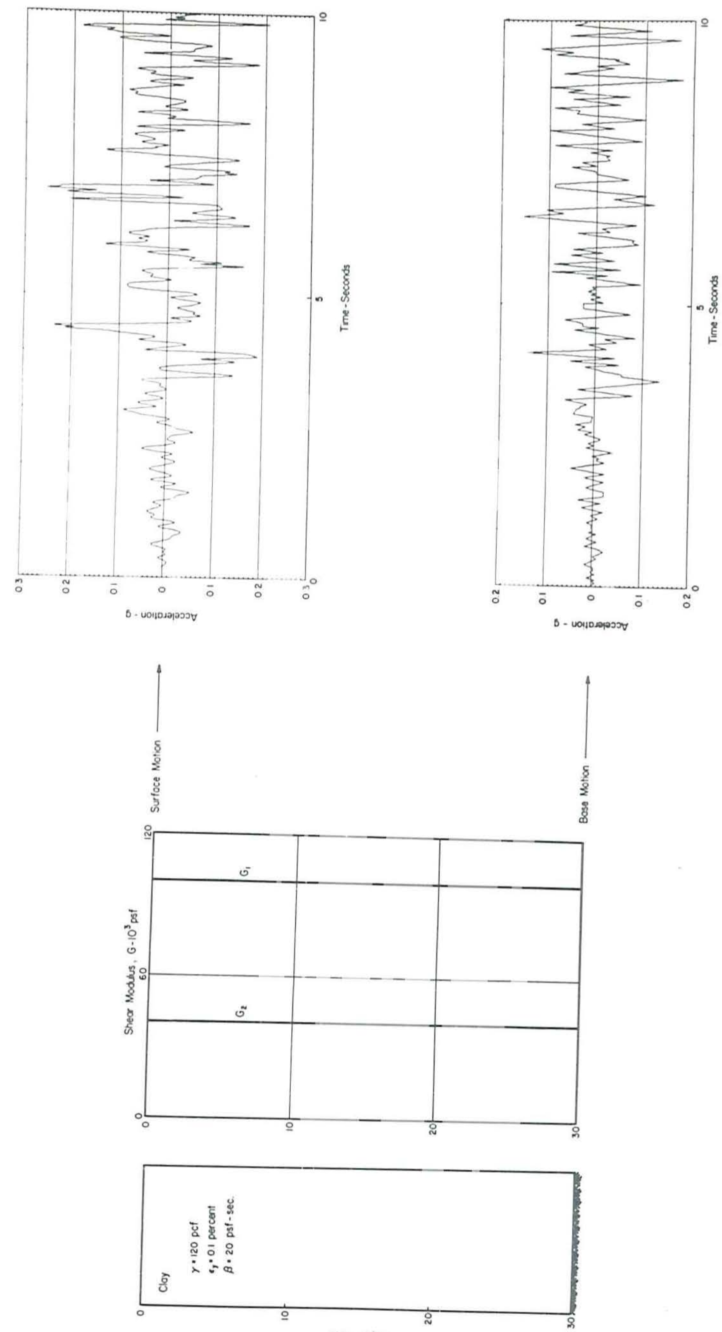

FIG. 13.—SURFACE RESPONSE OF LAYER WITH BILINEAR STRESS-STRAIN CHARACTERISTICS

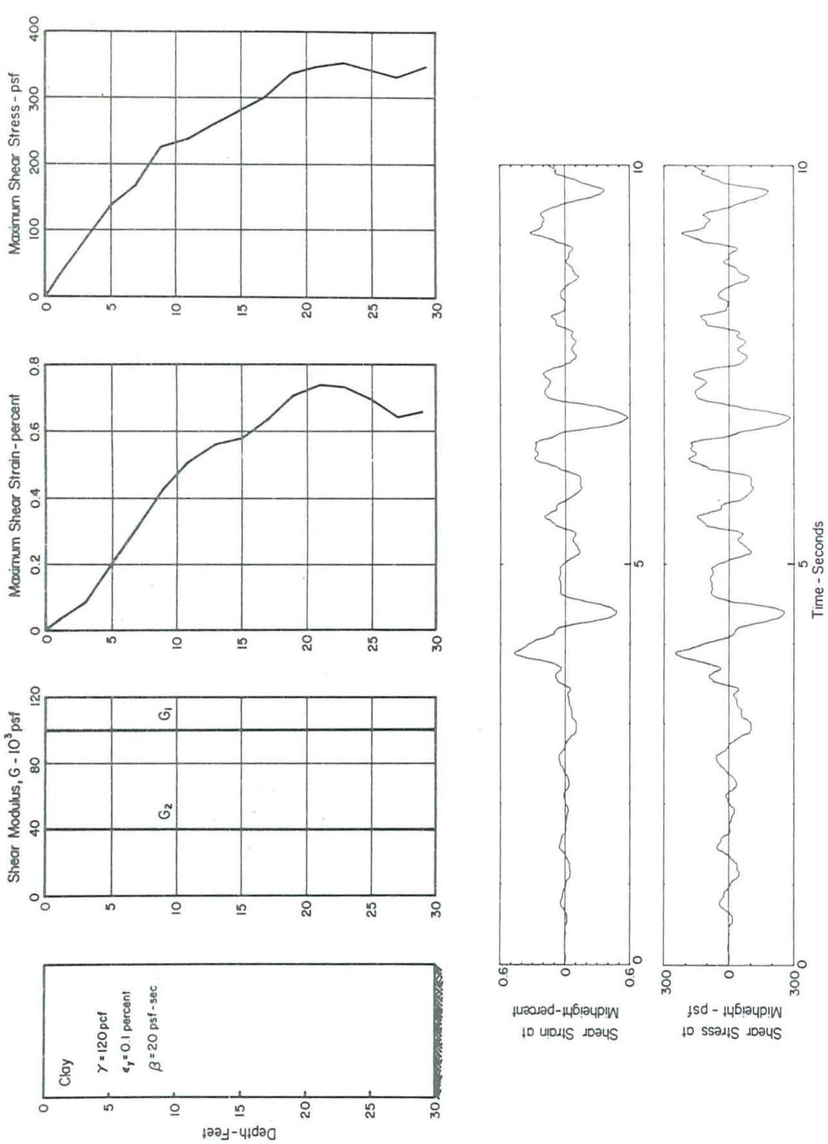

FIG. 14.—STRESSES AND STRAINS DEVELOPED WITHIN LAYER WITH BILINEAR STRESS-STRAIN CHARACTERISTICS

of the system (i.e., period of the 15th mode when $G = G_1$) is equal to 0.0384 sec.

The response of the layer was evaluated using the acceleration record shown in Fig. 13 as input base motion. This acceleration record is the first 10 sec

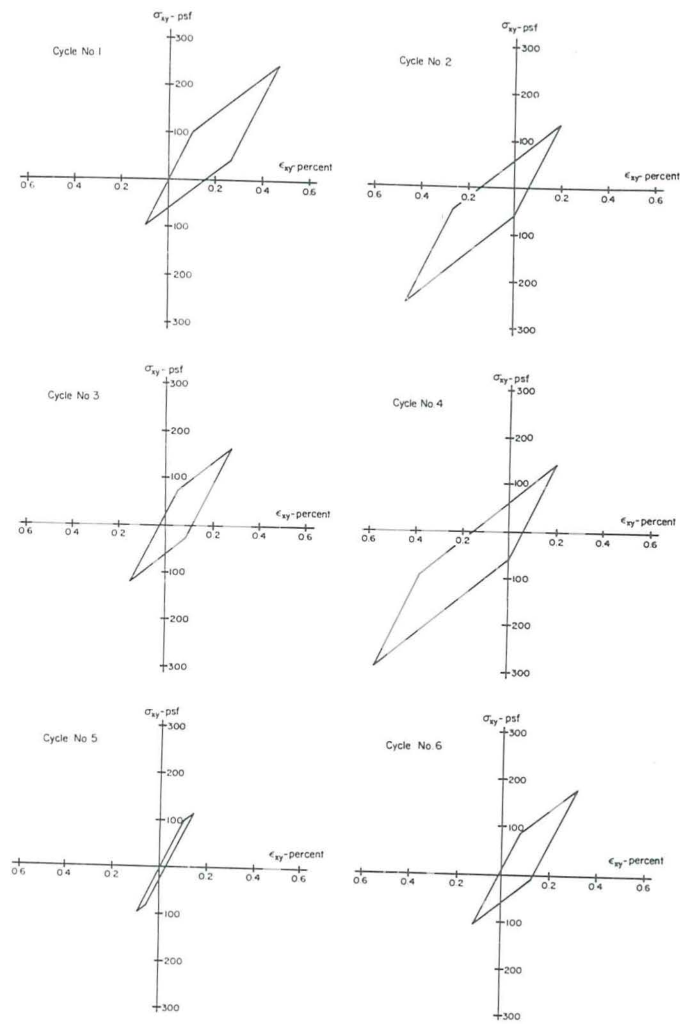

FIG. 15.—HYSTERETIC CYCLES DEVELOPED AT MIDHEIGHT OF LAYER WITH BILINEAR STRESS-STRAIN CHARACTERISTICS

of that recorded at Taft during the 1952 Kern County, California, earthquake. A time increment, Δt, equal to 0.01 sec was used in the numerical integration of the equations of motion to assure a stable solution ($T_{NN} > 2\Delta t$).

The time history of surface acceleration, evaluated for this layer using the bilinear solution, together with a trace of the input base motion are shown in Fig. 13. The maximum values of strain and stress developed throughout the depth of the layer and the time history of shear stress and shear strain at the midheight of the layer are presented in Fig. 14.

By selecting corresponding values of the shear stress and strain developed at different instants of time from the time history plots in Fig. 14, the cyclic nature of the stress-strain relationship developed in soil elements at the mid-height of the layer may be determined. The forms of the sequence of stress-strain cycles determined in this way are plotted in Fig. 15. The first complete cycle developed during the period of ground shaking from 0 sec to 4.25 sec; the second cycle was completed in the time interval from 4.26 sec to 5.90 sec. Altogether seven complete cycles were developed during the 10-sec period of ground shaking; six of these are shown in Fig. 15. These hysteretic cycles may be used to evaluate equivalent linear parameters that can be utilized in a linear elastic solution to evaluate the soil response as described in the following section.

EQUIVALENT LINEAR ANALYSIS

The use of an equivalent linear system to compute the response of a nonlinear system has been found to provide a reasonably satisfactory means of

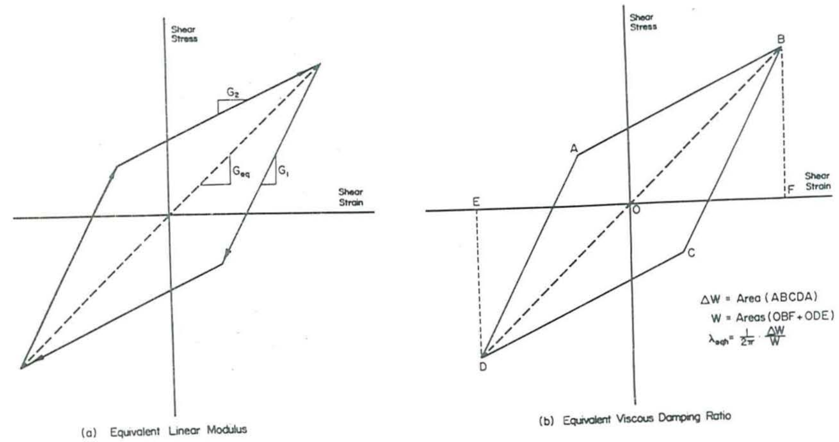

FIG. 16.—EVALUATION OF EQUIVALENT LINEAR PARAMETERS FROM BILINEAR HYSTERETIC STRESS-STRAIN CYCLE

evaluating dynamic behavior of single-degree-of-freedom systems. Jacobsen (12,13) and Hudson (9) characterized the effects of nonlinearities of a single-degree-of-freedom system by an equivalent viscous damping. Kryloff and Bogoliuboff (2,17) proposed the use of both an equivalent linear spring constant, and an equivalent damping ratio for a single-degree-of-freedom system having nonlinear characteristics. Caughy (3) extended the Kryloff-Bogoliuboff technique to the analysis of a nonlinear single-degree-of-freedom system with random excitation, and suggested means for obtaining the equivalent linear

spring constant and the equivalent viscous damping ratio to be used in the linear system. He proposed that the linear system, which gives the minimum mean squared difference between the differential equations of motion of the nonlinear and linear systems, be chosen as the equivalent linear system.

An equivalent linearization procedure may also be used to evaluate the seismic response of soil layers. The procedure involves the determination of an equivalent linear modulus, G_{eq}, and an equivalent damping ratio, λ_{eq}, for use in a linear elastic solution. For a single hysteretic stress-strain cycle, the value of G_{eq} may be taken as the chord modulus of the loop; i.e., the slope of the line joining the extreme points of the hysteresis loop as shown in Fig. 16(*a*). For a response involving a number of cycles of different stress and strain amplitudes, it would be appropriate to use the average value, $\overline{G}_{eq}$, of the values of G_{eq} corresponding to the different cycles.

Similarly an average value of the equivalent viscous damping ratio, $\overline{\lambda}_{eqh}$, corresponding to the hysteretic damping of the nonlinear system may be evaluated. For a single hysteretic stress-strain cycle the equivalent viscous damping λ_{eqh}, may be determined by the method originally proposed by Jacobsen (12) and illustrated in Fig. 16(*b*). For a response involving a number of different stress-strain cycles the average value $\overline{\lambda}_{eqh}$ would be adopted. It should be noted, however, that the equivalent total damping ratio, λ_{eq}, would be the sum of the average viscous damping ratio $\overline{\lambda}_{eqh}$ which is equivalent to the hysteretic damping in the bilinear system and the actual viscous damping, λ_{vis}, of the bilinear system even if the yield strain, ϵ_y, is not exceeded; i.e.,

$$\lambda_{eq} = \overline{\lambda}_{eqh} + \lambda_{vis}$$

The value of λ_{vis} would be determined directly from the viscous damping coefficient β, of the bilinear system.

The usefulness of this procedure may be illustrated by using it to compute the response of the 30-ft clay layer shown in Fig. 13, to the same base motion for which response values are presented in Figs. 13, 14, and 15.

Equivalent linear parameters may be determined from the stress-strain

TABLE 3.—EQUIVALENT LINEAR PARAMETERS FOR HYSTERETIC CYCLES OF BILINEAR SOLUTION

Cycle Number	Equivalent Linear Shear Modulus, G_{eq}, in pounds per square foot	Equivalent Viscous Damping Ratio, λ_{eqh}, as a percentage
1	61,000	13.5
2	58,000	12.7
3	66,000	12.7
4	55,000	11.2
5	90,000	7.5
6	68,000	12.1
7	60,000	13.3
	$\overline{G}_{eq}$ = 66,000	$\overline{\lambda}_{eqh}$ = 11.8

cycles for soil elements at the midheight of the layer shown in Fig. 15. Values of G_{eq} and λ_{eqh} for these cycles and a seventh cycle not shown in the figure are listed in Table 3, together with the average values, $\overline{G}_{eq}$ and $\overline{\lambda}_{eqh}$. The viscous damping ratio, λ_{vis}, of the bilinear system corresponding to the damping coefficient β = 20 psf-sec is about 6%. Thus appropriate values of the equivalent linear modulus, G_{eq}, and equivalent linear damping ratio, λ_{eq}, might be taken as

$$G_{eq} = \overline{G}_{eq} = 66{,}000 \text{ psf}$$

$$\lambda_{eq} = \overline{\lambda}_{eqh} + \lambda_{vis} = 11.8 + 6 = 17.8\%$$

Similar values for $\overline{G}_{eq}$ and $\overline{\lambda}_{eqh}$ might have been determined by computing the values of G_{eq} and λ_{eqh} for the hysteresis cycle corresponding to the average strain developed during the period of ground shaking at the midheight of the layer.

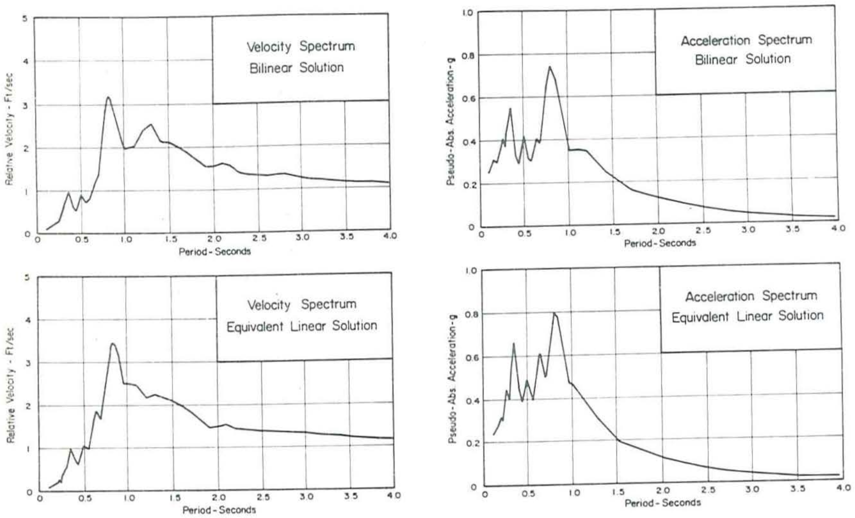

FIG. 17.—RESPONSE SPECTRA FOR BILINEAR AND EQUIVALENT LINEAR SOLUTIONS

The response of the 30-ft layer having a unit weight of 120 pcf, a shear modulus of 66,000 psf, and a damping ratio of 17.8% to the base acceleration record shown in Fig. 13, as determined by a linear lumped-mass analysis, is shown in Fig. 17 and Table 4. For computation purposes the layer was divided into 15 segments. The fundamental period of the layer was found to be 0.902 sec and the shortest period included in the analysis (i.e., period of the 15th mode) is 0.047 sec. The time interval, Δt, used in the numerical integration of the equation of motion was 0.01 sec, thereby assuring a stable solution with a high degree of accuracy.

The most important response values in the seismic analysis of soil layers are usually: (1) The surface accelerations, from which response spectra can be obtained; and (2) the shear stresses developed at various depths within the layer. For the 30-ft layer under discussion the maximum surface acceleration

obtained by the bilinear solution was 0.26 g and that determined by the equivalent linear solution was 0.24 g. The relative velocity and pseudo-absolute acceleration spectra for a single-degree-of-freedom structure with a damping ratio of 5%, obtained using the time history of surface accelerations of both solutions, are shown in Fig. 17. The maximum values of spectral velocity and acceleration are equal to 3.2 fps and 0.75 g for the bilinear solution. The corresponding maximum spectral values for the equivalent linear solution are equal to 3.5 fps and 0.8 g. The spectral intensity [i.e., the area under the relative velocity spectrum between a period of 0.1 sec to a period of 2.5 sec (8)] is equal to 3.94 ft for the spectrum of the bilinear solution, and that for the spectrum of the equivalent linear solution is equal to 4.08 ft. The maximum value of shear stresses obtained by the two solutions, within the upper 15 ft of the layer, are listed in Table 4. It may be seen from these results that the response values determined by the equivalent linear solution are in good agreement with those determined by the bilinear solution.

TABLE 4.—MAXIMUM VALUES OF SHEAR STRESSES DETERMINED BY BILINEAR AND EQUIVALENT LINEAR SOLUTIONS

Depth, in feet	Maximum Shear Stress, in pounds per square foot	
	Bilinear solution	Equivalent linear solution
1	31	26
3	86	78
5	145	131
7	172	167
9	245	215
11	273	251
13	297	283
15	301	309

Thus, it would appear that the response of a deposit, composed of essentially horizontal layers, to a horizontal excitation at the base can be made using a linear lumped-mass analysis incorporating appropriate values of the equivalent modulus and equivalent damping factor at the various depths. Such values can be determined readily from soil test data presented in the form of hysteresis loops corresponding to the average strain developed in a particular layer. While there is some analytical evidence to show that the applicability of the procedure may well be limited to cases where the bilinear moduli G_1 and G_2 are not very different, e.g., the ratio G_1/G_2 is not more than 4 or 5, available experimental results indicate that this condition is often satisfied in the bilinear representation of soil behavior. Thus, the use of this procedure provides a convenient approach for the analysis of many field problems.

SUMMARY AND CONCLUSIONS

Methods of analysis of the response of soil layers during earthquakes have been presented. These include linear elastic analyses, a bilinear analysis, and an equivalent linear analysis. All these methods require that: (1) The surface of the layer, the interface between any two sublayers and the base of the layer be essentially horizontal; (2) the material properties of the layer be constant along any horizontal plane; and (3) the applied seismic excitation be also horizontal. Variations from these conditions, however, can be readily handled using finite elements (10).

Both closed-form solutions and a lumped-mass representation were presented for the analysis of soil layers with linearly elastic properties. Once the geometry and material properties of the layer and the input base seismic motion are determined, these solutions provide response values (including accelerations, velocities, displacement, stresses and strains) throughout the layer for the duration of the input base motion. The closed-form solutions were utilized to evaluate the accuracy and stability of the lumped-mass analysis, and criteria for the accuracy and stability of the lumped-mass representation have been proposed.

A lumped-mass representation has also been used for the analysis of soil layers having bilinear stress-strain characteristics. Also response values can be calculated throughout these layers once the geometry and material properties of the layers and the applied base motion are known. Finally a procedure has been outlined for obtaining equivalent linear parameters for soils with bilinear stress-strain characteristics. The results obtained by this procedure have been shown to be in good agreement with those determined by the bilinear solution. This result offers considerable advantages for the evaluation of level ground response in engineering practice.

ACKNOWLEDGMENTS

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APPENDIX II.—NOTATION

b = constant as defined in Eq. 7;
$[C]$ = viscous damping matrix;
$c(y)$ = viscous damping coefficient at a depth y;
c_i^d = viscous damping coefficient of the Kelvin model connecting masses m_i and m_{i+1} as shown in Fig. 12;
E = modulus of elasticity;
G_{eq} = equivalent linear shear modulus as shown in Fig. 16;
$G(y)$ = shear modulus at depth y;
G_1 = initial or primary shear modulus of the bilinear system as shown in Fig. 11;
G_2 = secondary shear modulus of the bilinear system as shown in Fig. 11;
H = total thickness of a semi-infinite deposit as shown in Figs. 1, 7, and 12;
$2h_i$ = thickness of the segment of the deposit between level i and level $i + 1$ as shown in Figs. 7 and 12;
i = index;
J_{-b} = Bessel function of the first kind of order $-b$;
K = constant as defined in Eq. 2;
$[K]$ = stiffness matrix;
k_1 = initial or primary spring stiffness coefficient of the Kelvin model;
k_2 = secondary spring stiffness coefficient of the Kelvin model;
$[M]$ = mass matrix;
m_i = lumped mass of level i;
N = number of levels into which a semi-infinite deposit is divided;
n = index;
p = constant as defined in Eq. 2;
R_n = modal participation factor as defined in Eq. 6;
$\{R(t)\}$ = earthquake load vector;
T_1 = fundamental period of a semi-infinite layer determined by a closed-form solution;
T_{1N} = fundamental period of a semi-infinite deposit determined by a lumped-mass solution;
T_{NN} = period of the highest mode of vibration included in the analysis;
t = time ordinate;
$\{u\}$ = relative displacement vector;
u_g = displacement at base of a semi-infinite deposit;
u_i = relative displacement of level i;
u_i^s = displacement across Kelvin model as shown in Fig. 12;
u_{y_i} = yield displacement of the bilinear system;
$u(y, t)$ = relative displacement at depth y at time t;
X_n = normal coordinate for n^{th} mode of vibration;
$\{X(t)\}$ = normal coordinates vector;
$Y_n(y)$ = n^{th} mode shape at a depth y of a semi-infinite layer determined by a closed-form solution;

y = depth ordinate;
β_i = viscous damping coefficient as defined in Eq. 14;
β_n = roots of Bessel function $J_{-b}(\beta_n) = 0$;
Γ = gamma function;
γ = unit weight;
ΔF_i = change in force across Kelvin model of bilinear system;
Δt = time increment;
ϵ_{xy} = shear strain;
ϵ_{yi} = yield shear strain;
θ = constant as defined in Eq. 7;
λ_{eq} = equivalent linear damping ratio;
λ_{eqh} = equivalent hysteretic damping ratio as shown in Fig. 16;
λ_n = damping ratio for n^{th} mode of vibration;
λ_{vis} = viscous damping ratio;
μ = Poisson's ratio;
ρ = mass density;
σ_{xy} = shear stress;
$\{\phi^n\}$ = vector of mode shape for n^{th} mode of vibration; and
ω_n = circular frequency of n^{th} mode of vibration.